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A note on global existence of weak solutions to the compressible magnetohydrodynamic equations with Coulomb force

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ABSTRACT

In this note, for the case of $(9+\sqrt{33})/12 < \gamma \leqslant 4/3$, we prove the existence of global-in-time finite energy weak solution of the equations of a two-dimensional magnetohydrodynamics with Coulomb force, where γ denotes the adiabatic exponent. The value $(9+\sqrt{33})/12$ is the optimal lower bound of γ to establish global-in-time finite energy weak solution under current frame.

1. Introduction

The magnetohydrodynamic equations can be written as, in non-dimensional form (cf. [1–3]):

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(p + \frac{1}{2}|\mathbf{M}|^2\right) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} = \operatorname{div}(\mathbf{M}\otimes\mathbf{M}) + \rho\nabla\Phi, \tag{1.2}$$

$$\mathbf{M}_t + \operatorname{div}(\mathbf{u}\otimes\mathbf{M}) - \operatorname{div}(\mathbf{M}\otimes\mathbf{u}) - \nu\Delta\mathbf{M} = 0, \qquad \operatorname{div}\mathbf{M} = 0, \tag{1.3}$$

$$\Delta\Phi = \rho - \bar{\rho}. \tag{1.4}$$

This system can be used to describe, for example, the dynamics of a charge transport where the compressible electron fluid interacts with its own electric field against a charged ion background together under the influence by the magnetic field, in which ρ, $\mathbf{u}$, Φ and $\mathbf{M}$ represent the electron density, electron velocity, electrostatic potential and magnetic field respectively. $\bar{\rho} > 0$ is the constant background ion density. $p = p(\rho) = a\rho^{\gamma}$ is the pressure with the positive constant $a > 0$ and the adiabatic exponent $\gamma > 1$. The constant viscosity coefficients μ, λ satisfy $\mu > 0$ and $\lambda + 2\mu/N \geqslant 0$, where N is dimension.

By neglecting the displacement current, the third equation is the induction equation concerning the magnetic field $\mathbf{M}$ and $\nu > 0$ in terms of the magnetic diffusivity of the fluid. The motion of such fluids is driven by two dominating body forces, namely, the Coulomb force (i.e. the electric field force) and the Lorentz force imposed on the fluid by the magnetic field. This makes the physical phenomena of such fluids much more complicated.

The initial–boundary value problem for the system (1.1)–(1.4) is studied recently by [4–6] concerning the existence of weak solutions. These works imply that for $\gamma > 3/2$ and when the dimension is three the system (1.1)–(1.4) admits a global

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weak solution for the bounded smooth domain Ω. On the other hand, in the well-known framework of the compressible Navier–Stokes equations [7,8], the existence of globally defined finite energy weak solutions was proved under the somehow optimal constrain $\gamma > N/2$, where N is the dimension. This is justified for the system (1.1)–(1.4), as we quoted before, when the dimension is three. The natural question is to ask that whether it is also true for the two-dimensional case. The author in [9] gave the positive results. Before further remarking their results in [9], let us recall the definition of the globally defined finite energy weak solutions for reader's convenience.

We consider the system (1.1)–(1.4) in a two-dimensional bounded smooth domain, i.e., for $(t,\mathbf{x}) \in I \times \Omega$, $\Omega \subset \mathbb{R}^2$ is bounded and smooth, and we prescribe the following initial–boundary conditions

$$\begin{aligned}&(\rho, \rho\mathbf{u}, \mathbf{M})|_{t=0} = (\rho_0, \mathbf{q}_0, \mathbf{M}_0) \quad \text{in } \Omega,\\ &\mathbf{u} = \mathbf{0}, \qquad \mathbf{M} = 0, \qquad \nabla\Phi \cdot \mathbf{n} = 0 \quad \text{on } I \times \partial\Omega,\end{aligned} \tag{1.5}$$

where $\mathbf{n}$ denotes the outer normal vector of the boundary of Ω and $I = (0, T)$.

Definition 1.1. We call $(\rho, \mathbf{u}, \Phi, \mathbf{M})$ a finite energy weak solution of (1.1)–(1.5) if:

- $(\varrho, \mathbf{u}, \Phi, \mathbf{M})$ belongs to the following class

$$\begin{aligned}&0 \leqslant \rho \in L^\infty\big(I, L^\gamma(\Omega)\big), \qquad \mathbf{u} \in L^2\big(I, \big(H_0^1(\Omega)\big)^2\big), \qquad \Phi \in L^\infty\big(I, H^1(\Omega)\big),\\ &\mathbf{M} \in L^\infty\big(I, \big(L^2(\Omega)\big)^2\big) \cap L^2\big(I, \big(H_0^1(\Omega)\big)^2\big), \qquad \operatorname{div}\mathbf{M}(t) = 0 \quad \text{for a.a. on } t \in I.\end{aligned} \tag{1.6}$$

- Let $E(t)$ be the total energy defined as

$$E(t) = \int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\rho^\gamma + \frac{1}{2}|\mathbf{M}|^2 + \frac{1}{2}|\nabla\Phi|^2\,\mathrm{d}\mathbf{x}, \tag{1.7}$$

then $E = E(t) \in L^1_{\mathrm{loc}}(I)$ satisfies the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \int_\Omega \mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\operatorname{div}\mathbf{u}|^2 + \nu|\nabla\mathbf{M}|^2\,\mathrm{d}\mathbf{x} \leqslant 0 \quad \text{in } \mathcal{D}'(I). \tag{1.8}$$

- For any $t \in I$, the pair (ρ, Φ) satisfies the following Poisson equation

$$\Delta\Phi = \rho - \bar\rho \quad \text{in } \Omega, \qquad \nabla\Phi \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad \text{in the sense of traces}, \qquad \int_\Omega \Phi\,\mathrm{d}\mathbf{x} = 0. \tag{1.9}$$

- Eq. (1.1) holds in $\mathcal{D}'([0,T) \times \mathbb{R}^2)$, i.e.,

$$\int_I\int_\Omega (\varrho\varphi_t + \varrho\mathbf{u}\cdot\nabla\varphi)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega \varrho_0\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \tag{1.10}$$

for any $\varphi \in \mathcal{D}([0,T) \times \mathbb{R}^2)$. Moreover, the pair $(\rho, \mathbf{u})$ satisfies the renormalized equation

$$\int_I\int_\Omega \big[b(\rho)\varphi_t + b(\rho)\mathbf{u}\cdot\nabla\varphi + \big(b(\rho) - b'(\rho)\rho\big)\operatorname{div}\mathbf{u}\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_\Omega b(\rho_0)\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{1.11}$$

for any function $\varphi \in \mathcal{D}([0,T) \times \mathbb{R}^2)$ and $b \in C^1(\mathbb{R})$ such that

$$b'(z) = 0 \quad \text{for} |z| \text{ large enough.} \tag{1.12}$$

- Eqs. (1.2), (1.3) hold in $(\mathcal{D}'([0,T) \times \Omega))^2$, i.e.,

$$\begin{aligned}&\int_I\int_\Omega \left(\varrho\mathbf{u}\cdot\varphi_t + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\varphi + p\operatorname{div}\varphi + \frac{1}{2}|\mathbf{M}|^2\operatorname{div}\varphi\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\ &\quad = -\int_\Omega \mathbf{q}\cdot\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_I\int_\Omega \big[\mu\nabla\mathbf{u} : \nabla\varphi + (\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\ &\qquad + \int_I\int_\Omega \mathbf{M}\otimes\mathbf{M} : \nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_I \langle\varrho\nabla\Phi, \varphi\rangle\,\mathrm{d}t,\end{aligned} \tag{1.13}$$

$$\int_I\int_\Omega \big[\mathbf{M}\cdot\varphi_t + (\mathbf{u}\otimes\mathbf{M} - \mathbf{M}\otimes\mathbf{u}) : \nabla\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_\Omega \mathbf{M}_0\cdot\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_I\int_\Omega \nu\nabla\mathbf{M} : \nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t, \tag{1.14}$$

for any $\varphi \in (\mathcal{D}([0,T)\times\Omega))^2$, where the dual product $\langle \varrho\nabla\Phi, \varphi\rangle \in L^1(I)$ is defined as

$$\langle \rho\nabla\Phi, \varphi\rangle = \int_\Omega \rho\nabla\Phi\varphi\,\mathrm{d}\mathbf{x}. \tag{1.15}$$

In the above definition the regularity of ρ in (1.6) comes from the well-known energy inequality. It immediately implies $\Phi \in L^\infty(I, W^{2,\gamma}(\Omega))$ by the elliptic regularity of Eq. (1.4). Hence the Sobolev's inequality implies $\nabla\Phi \in L^\infty(I, (L^{\frac{2\gamma}{2-\gamma}}(\Omega))^2)$. We see that $\rho\nabla\Phi \in L^\infty(I, (L^{\frac{2\gamma}{4-\gamma}}(\Omega))^2)$ for the case $4 > \gamma \geqslant 4/3$, where $\frac{2\gamma}{4-\gamma} \geqslant 1$. Thus the existence of the finite energy weak solution of the problem (1.1)–(1.5) can be easily verified by the frame in [7] without any modifications for the case of $\gamma \geqslant \frac{4}{3}$. But when the adiabatic exponent γ close to one, i.e. $1 < \gamma < 4/3$, one cannot even have $\rho\nabla\Phi$ be integral, i.e., whether the $\int_0^T \langle \rho\nabla\Phi, \varphi\rangle\,\mathrm{d}t = \int_0^T\int_\Omega \rho\nabla\Phi\varphi\,\mathrm{d}x\,\mathrm{d}t$ in (1.13) is finite. For overcoming this difficulty, the authors in [9] make use of the Poisson equation (1.4) to regard $\rho\nabla\Phi$ as a generalized function (cf. [9]) and renewed to definite so-called generalized finite energy weak solutions.

Definition 1.2. We call $(\rho, \mathrm{u}, \Phi, \mathbf{M})$ a generalized finite energy weak solution of (1.1)–(1.5) if $(\rho, \mathrm{u}, \Phi, \mathbf{M})$ satisfies (1.6)–(1.14), where $\langle \rho\nabla\Phi, \varphi\rangle$ in (1.13) is defined as

$$\begin{aligned}
\langle \rho\partial_{x_i}\Phi, \varphi\rangle &:= \langle \rho\partial_{x_i}\Phi, \varphi\rangle_{W^{-1,\frac{\gamma}{2-\gamma}}(\Omega)\times W_0^{1,\frac{\gamma}{2\gamma-2}}(\Omega)} \\
&= \big\langle (\rho-\bar\rho)\partial_{x_i}\Phi, \varphi\big\rangle + \bar\rho\langle \partial_{x_i}\Phi, \varphi\rangle \\
&= \langle \Delta\Phi\partial_{x_i}\Phi, \varphi\rangle + \bar\rho\int_\Omega \partial_{x_i}\Phi\varphi\,\mathrm{d}\mathbf{x} \\
&:= \int_\Omega \frac{1}{2}|\nabla\Phi|^2\partial_{x_i}\varphi\,\mathrm{d}\mathbf{x} - \int_\Omega \partial_{x_i}\Phi\nabla\Phi\cdot\nabla\varphi\,\mathrm{d}\mathbf{x} + \bar\rho\int_\Omega \partial_{x_i}\Phi\varphi\,\mathrm{d}\mathbf{x},
\end{aligned} \tag{1.16}$$

$i = 1, 2$ and

$$\rho\nabla\Phi \in L^\infty\big(I, \big(W^{-1,\frac{\gamma}{2-\gamma}}(\Omega)\big)^2\big). \tag{1.17}$$

Thus the authors in [9] established the following result.

Proposition 1.1. *Assume $\Omega \subset \mathbb{R}^2$ is a smooth domain, let $1 < \gamma \leqslant \frac{4}{3}$ and the initial data satisfy the compatibility conditions*

$$\begin{aligned}
&\rho_0 \in L^\gamma(\Omega), \qquad \rho_0 \geqslant 0, \qquad \int_\Omega (\rho_0 - \bar\rho)\,\mathrm{d}\mathbf{x} = 0, \qquad \mathbf{q}_0(\mathbf{x}) = \mathbf{0} \quad \textit{whenever } \rho_0(\mathbf{x}) = 0, \\
&\frac{|\mathbf{q}_0|^2}{\rho_0} \in L^1(\Omega), \qquad \mathbf{M}_0 \in L^2(\Omega) \quad \textit{with } \operatorname{div}\mathbf{M}_0 = 0 \textit{ in } \mathcal{D}'(\Omega).
\end{aligned} \tag{1.18}$$

Then for any given $T > 0$, there exists a global-in-time generalized finite energy weak solution $(\rho, \mathbf{u}, \Phi, \mathbf{M})$ of the problem (1.1)–(1.5).

Remark 1.1. The initial conditions in (1.18) make sense, please refer to [9, Remark 1.1] for details.

The natural question is to ask whether the definition (1.16) for the generalized function $\varrho\nabla\Phi$ in Proposition 1.1 can be replaced by (1.15). Here we obtain the positive result for the case $(9+\sqrt{33})/12 \leqslant \gamma \leqslant 4/3$ and give the proof of our result by Proposition 1.1 in next section.

2. Results and proofs

After previous introduction, we start to establish our result.

2.1. Computations for adiabatic γ and results

From Proposition 1.1, we see that ρ belongs to $L^\infty(I, L^\gamma(\Omega))$. But we find that ρ have more higher integrability from the proof of Proposition 1.1. In fact, due to [9, Lemma 3.2], we have

$$\rho \in L^{\gamma+\theta}(Q_T), \tag{2.1}$$

where $Q_T = I \times \Omega$ and $0 < \theta < \gamma - 1$ for the case of $1 < \gamma \leqslant \frac{4}{3}$. It immediately implies $\Phi \in L^{\gamma+\theta}(I, W^{2,\gamma+\theta}(\Omega))$ by the elliptic regularity of Eq. (1.14), thus $\nabla\Phi \in L^{\gamma+\theta}(I, (L^{\frac{2(\gamma+\theta)}{2-(\theta+\gamma)}}(\Omega))^2)$ by Sobolev's inequality. So we get that $\rho \in L^{\gamma+\theta}(Q_T) \cap L^\infty(I, L^\gamma(\Omega))$ and $\nabla\Phi \in L^{\gamma+\theta}(I, (L^{\frac{2(\gamma+\theta)}{2-(\theta+\gamma)}}(\Omega))^2) \cap L^\infty(I, (L^{\frac{2\gamma}{2-\gamma}}(\Omega))^2)$.

By computation, we find that

$$\frac{1}{2\gamma-1} + \frac{2-\gamma}{2\gamma} < 1 \quad \Leftrightarrow \quad 6\gamma^2 - 9\gamma + 2 > 0 \quad \Leftrightarrow \quad \frac{1}{\gamma} + \frac{2-(2\gamma-1)}{2(2\gamma-1)} < 1, \tag{2.2}$$

where $\frac{9+\sqrt{33}}{12} < \gamma \leqslant \frac{4}{3}$. This yields

$$\rho\nabla\Phi \in L^{\gamma+\theta}\big(I, \big(L^p(\Omega)\big)^2\big), \tag{2.3}$$

where $p = \frac{2\gamma(\theta+\gamma)}{(2-\gamma)(\theta+\gamma)+2\gamma}$, θ satisfies $\frac{2\gamma(\theta+\gamma)}{(2-\gamma)(\theta+\gamma)+2\gamma} \geqslant 1$ and $0 < \theta < \gamma - 1$ for the case of $\frac{9+\sqrt{33}}{12} < \gamma \leqslant \frac{4}{3}$. We note that $\frac{4}{3} > \frac{9+\sqrt{33}}{12} \approx 1.23$.

By above analysis and the frame of proof of [9, Theorem 1.1], we immediately obtain the following theorem.

Theorem 2.1. *Assume $\Omega \subset \mathbb{R}^2$ is a smooth domain, let $\frac{9+\sqrt{33}}{12} < \gamma \leqslant \frac{4}{3}$, then for any given $T > 0$, there exists a global-in-time finite energy weak solution $(\rho, \mathbf{u}, \Phi, \mathbf{M})$ of the problem (1.1)–(1.5), which is such that*

$$\rho \in L^{\gamma+\theta}(Q_T) \quad \text{and} \quad \rho\nabla\Phi \in L^{\gamma+\theta}\big(I, L^p(\Omega)\big), \tag{2.4}$$

where $0 < \theta < \gamma - 1$ and $p = \frac{2\gamma(\theta+\gamma)}{(2-\gamma)(\theta+\gamma)+2\gamma} \geqslant 1$.

Remark 2.1. We remark that, as for a particular case when magnetic fluid $\mathbf{M}$ is absent, above Theorem 2.1 also implies the global existence of the finite energy weak solutions to the Navier–Stokes-Poisson equations in [10–12] for two-dimensional case.

Remark 2.2. Of course, Theorem 2.1 also holds for $\gamma > 4/3$.

2.2. Proof of Theorem 2.1 by Proposition 1.1

Though we can get Theorem 2.1 under the frame of [9, Theorem 1.1], here we note that Theorem 2.1 also can be inferred from Proposition 1.1. We can complete this proof by the following three lemmas, the main idea is to prove (1.15) and (1.16) are equivalent under Proposition 1.1.

Lemma 2.1. *Let $(\varrho, \mathbf{u})$ satisfy the regularity (1.6) and the renormalized equation (1.11), where $b \in C^1(\mathbb{R})$ satisfies (1.12), ϱ and $\mathbf{u}$ are zero prolong out Ω.*

Then the renormalized equation (1.11) also holds for b being relaxed to satisfy

$$b \in C[0,\infty) \cap C^1(0,\infty), \qquad \big|b'(t)\big| \leqslant Ct^{-\lambda_0}, \quad t \in (0,1),\ \lambda_0 < 1, \tag{2.5}$$

and

$$\big|b'(t)\big| \leqslant Ct^{\lambda_1}, \quad t \geqslant 1,\ C > 0,\ -1 < \lambda_1 \leqslant \frac{\theta(\gamma)}{2} - 1, \tag{2.6}$$

where C is some constant only dependent on b and $\theta(\gamma) = \gamma$.

In particular, for any $\varphi \in \mathcal{D}([0,T) \times \mathbb{R}^2)$, we have

$$\int_0^T \int_\Omega \big[b_k(\varrho)\varphi_t + b_k(\varrho)\mathbf{u}\cdot\nabla\varphi + \big(b_k(\varrho) - (b_k)'_+(\varrho)\varrho\big)\operatorname{div}\mathbf{u}\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_\Omega b_k(\varrho_0)\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{2.7}$$

where

$$b_k(t) = \begin{cases} b(t), & \text{if } 0 \leqslant t < k, \\ b(k), & \text{if } t \geqslant k, \end{cases} \tag{2.8}$$

b satisfies (2.5)–(2.6) and

$$(b_k)'_+(t) = \begin{cases} b'(t), & \text{if } 0 \leqslant t < k, \\ 0, & \text{if } t \geqslant k. \end{cases} \tag{2.9}$$

Proof. Let $b(t)$ satisfy (2.5)–(2.6) and $b(t)$ be prolonged by $b(0)$ outside $\mathbb{R}_0^+$. For $k>0$, by the definition of $b_k(t)$, we get

$$b_k' \in C\big([0,k)\cup(k,+\infty)\big), \qquad \lim_{t\to k^-} b_k'(t) = b'(k), \qquad \lim_{t\to k^+} b_k'(t) = 0.$$

We see that the right derivative $(b_k)'_+(t)$ exists on $[0,+\infty)$ and

$$\big[(b_k)'_+\circ\varrho\big](\mathbf{x}) = \begin{cases} b_k'(\varrho(\mathbf{x})), & \text{if } \mathbf{x}\in\{\varrho\neq k\},\\ 0, & \text{if } \mathbf{x}\in\{\varrho=k\}.\end{cases} \tag{2.10}$$

We take $d\in\mathcal{D}(\mathbb{R})$ such that $\operatorname{supp} d\subset(0,+\infty)$, $d(t)=t$ on $(3k/4,5k/4)$, and then set $d^+_{k,\varepsilon}=S_{\varepsilon/2}(d_{k+\varepsilon})$ and $d^-_{k,\varepsilon}=S_{\varepsilon/2}(d_{k-\varepsilon})$, where S_ε is the standard one-dimension mollifier and d_k is defined as (2.8), we have, as $\varepsilon\to0^+$,

$$\begin{aligned}
&\forall t\in\mathbb{R}_+,\quad d^+_{k,\varepsilon}(t)\to d_k(t); \qquad \forall t\neq k,\quad \big[d^+_{k,\varepsilon}\big]'(t)\to d_k'(t); \qquad \big[d^+_{k,\varepsilon}\big]'(k)\to 1;\\
&\forall t\in\mathbb{R}_+,\quad d^-_{k,\varepsilon}(t)\to d_k(t); \qquad \forall t\neq k,\quad \big[d^-_{k,\varepsilon}\big]'(t)\to d_k'(t); \qquad \big[d^-_{k,\varepsilon}\big]'(k)\to 0.
\end{aligned}$$

Using (1.11), we have

$$\int_0^T\int_\Omega \big\{d^\pm_{k,\varepsilon}(\varrho)\varphi_t + d^\pm_{k,\varepsilon}(\varrho)\mathbf{u}\cdot\nabla\varphi + \big[d^\pm_{k,\varepsilon}(\varrho) - \big(d^\pm_{k,\varepsilon}\big)'(\varrho)\varrho\big]\operatorname{div}\mathbf{u}\varphi\big\}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_\Omega d^\pm_{k,\varepsilon}(\varrho_0)\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x},$$

for any $\varphi\in\mathcal{D}([0,T)\times\mathbb{R}^2)$. Therefore, letting $\varepsilon\to0^+$, we have

$$\int_0^T\int_\Omega \big[d_k(\varrho)\varphi_t + d_k(\varrho)\mathbf{u}\cdot\nabla\varphi + \big(d_k(\varrho) - d_k'(\varrho)1_{\{\varrho\neq k\}}\varrho - k1_{\{\varrho=k\}}\big)\operatorname{div}\mathbf{u}\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_\Omega d_k(\varrho_0)\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x},$$

and

$$\int_0^T\int_\Omega \big[d_k(\varrho)\varphi_t + d_k(\varrho)\mathbf{u}\cdot\nabla\varphi + \big(d_k(\varrho) - d_k'(\varrho)1_{\{\varrho\neq k\}}\varrho\big)\operatorname{div}\mathbf{u}\varphi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_\Omega d_k(\varrho_0)\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Subtracting the last two equations, since the $\varphi\in\mathcal{D}([0,T)\times\mathbb{R}^2)$ is arbitrary, we obtain

$$k\operatorname{div}\mathbf{u}=0 \quad \text{a.e. in } \{\varrho=k\}. \tag{2.11}$$

(1.11) applied to $S_\varepsilon[b_k]$ (b_k is prolonged by $b(0)$ outside $\mathbb{R}_0^+$) gives

$$\begin{aligned}
&\int_0^T\int_\Omega \big\{\big\{S_\varepsilon[b_k]\big\}(\varrho)\varphi_t + \big\{S_\varepsilon[b_k]\big\}(\varrho)\mathbf{u}\cdot\nabla\varphi + \big\{\big\{S_\varepsilon[b_k]\big\}(\varrho) - \big\{S_\varepsilon[b_k]\big\}'(\varrho)\varrho\big\}\operatorname{div}\mathbf{u}\varphi\big\}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad= -\int_\Omega \big\{S_\varepsilon[b_k]\big\}(\varrho_0)\varphi(0,\mathbf{x})\,\mathrm{d}\mathbf{x}.
\end{aligned}\tag{2.12}$$

Letting $\varepsilon\to0^+$, we have

$$\begin{aligned}
&\forall t\in[0,+\infty),\quad \big\{S_\varepsilon[b_k]\big\}(t)\to b_k(t); \qquad t=0,\quad t\big\{S_\varepsilon[b_k]\big\}'(t)\to 0;\\
&\forall t\in[0,k)\cup(k,+\infty),\quad \big\{S_\varepsilon[b_k]\big\}'(t)\to b_k'(t).
\end{aligned}\tag{2.13}$$

In accordance with (2.10), there holds $\{S_\varepsilon[b_k]\}(\varrho)\to b_k(\varrho)$ a.e. in $\mathbb{R}^2$, $\{S_\varepsilon[b_k]\}'(\varrho)\to b_k'(\varrho)$ a.e. in $\{\varrho\neq k\}$, $\varrho\{S_\varepsilon[b_k]\}'(\varrho)\to\varrho b_k'(\varrho)$ a.e. in $\{\varrho=0\}$. Due to (2.11), $\varrho\{S_\varepsilon[b_k]\}'(\varrho)\operatorname{div}\mathbf{u}=0$ a.e. in $\{\varrho=k\}$. Moreover, $|\{S_\varepsilon[b_k]\}(t)|$ and $|\{S_\varepsilon[b_k]\}'(t)|$ are uniformly bounded with respect to $\varepsilon>0$. We pass to the limit $\varepsilon\to0^+$ in (2.12), by Lebesgue dominated convergence theorem and we obtain that (2.7), let $k\to\infty$ in (2.7), we can get the desired conclusion. □

Lemma 2.2. *If* $(\varrho,\mathbf{u},\Phi,\mathbf{M})$ *is the generalized finite energy weak solution of equations of* (1.1)–(1.5)*, then*

$$\int_{Q_T}\varrho^{\gamma+\theta}\,\mathrm{d}\mathbf{x}\leqslant C, \tag{2.14}$$

where $0<\theta<\gamma-1$ *and* $1<\gamma\leqslant4/3$.

Proof. We use the mollifier S_ε to regularize the variable t in Eq. (2.7). For any fixed open interval I', $I' \subset I$ and for any $0 < \varepsilon < \varepsilon_0(I')$, where $\varepsilon_0(I')$ is sufficiently small (we can choose, e.g., $\varepsilon_0(I') = \min\{\sup I - \sup I', \inf I - \inf I'\}$), we get, for any $\varphi \in \mathcal{D}(I' \times \mathbb{R}^2)$,

$$\int_{I'}\int_{\Omega} \{S_\varepsilon[b_k]\}(\varrho)\varphi_t + \{S_\varepsilon[b_k]\}(\varrho)\mathbf{u}\cdot\nabla\varphi + S_\varepsilon\{[b_k(\varrho) - (b_k)'_+(\varrho)\varrho]\operatorname{div}\mathbf{u}\}\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0, \tag{2.15}$$

and take

$$b_k(t) = \left\{\begin{matrix} t^\theta, & \text{if } 0 \leqslant t < k, \\ k^\theta, & \text{if } t \geqslant k, \end{matrix}\right\} \leqslant t^\theta, \quad 0 < \theta < \gamma - 1,\ 1 < \gamma \leqslant 4/3. \tag{2.16}$$

By virtue of [13, Lemma 6.5] and the regularity (1.6), we have

$$\begin{aligned} &S_\varepsilon[b_k(\varrho)\mathbf{u}] \in C^\infty(I', (L^q(\mathbb{R}^2))^2), \quad 1 \leqslant q < +\infty, \qquad S_\varepsilon[b_k(\varrho)], \partial_t S_\varepsilon[b_k(\varrho)] \in C^\infty(I', L^\infty(\mathbb{R}^2)), \\ &S_\varepsilon\{[(b_k)'_+(\varrho)\varrho - b_k(\varrho)]\operatorname{div}\mathbf{u}\} \in C^\infty(I', L^2(\mathbb{R}^2)), \qquad \operatorname{div} S_\varepsilon[b_k(\varrho)\mathbf{u}] \in C^\infty(I', L^2(\mathbb{R}^2)), \end{aligned}$$

and $S_\varepsilon[b_k(\varrho)\mathbf{u}] \in C^\infty(I', E_0^2(\Omega))$, where

$$E_0^2(\Omega) = \{\mathbf{g} \in (L^2(\Omega))^2 \mid \operatorname{div}\mathbf{g} \in L^2(\Omega),\ \mathbf{g}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of trace}\}.$$

We define

$$\varphi(t,\mathbf{x}) = \psi(t)\phi(t,\mathbf{x}), \qquad \phi = \mathcal{B}\left(S_\varepsilon[b_k(\varrho)] - \frac{1}{|\Omega|}\int_\Omega S_\varepsilon[b_k(\varrho)]\,\mathrm{d}\mathbf{x}\right), \quad \psi \in \mathcal{D}(I), \tag{2.17}$$

where $\mathcal{B}$ is Bogovikii operator (please refer to [13, Lemma 3.17] for the property), and denote $\phi = (\phi^1, \phi^2)$. By standard density approximation, we see that $\varphi(t,\mathbf{x})$ can be testing function for (1.13). By computation, we have

$$\begin{aligned} \partial_t\varphi &= \psi'\phi(t,\mathbf{x}) + \psi\phi'(t,\mathbf{x}) \\ &= \psi'\mathcal{B}\left(S_\varepsilon[b_k(\varrho)] - \frac{1}{|\Omega|}\int_\Omega S_\varepsilon[b_k(\varrho)]\,\mathrm{d}\mathbf{x}\right) + \psi\mathcal{B}\left(\partial_t S_\varepsilon[b_k(\varrho)] - \frac{1}{|\Omega|}\int_\Omega \partial_t S_\varepsilon[b_k(\varrho)]\,\mathrm{d}\mathbf{x}\right) \\ &= \psi'\mathcal{B}\left(S_\varepsilon[b_k(\varrho)] - \frac{1}{|\Omega|}\int_\Omega S_\varepsilon[b_k(\varrho)]\,\mathrm{d}\mathbf{x}\right) - \psi\mathcal{B}(\operatorname{div} S_\varepsilon[b_k(\varrho)\mathbf{u}]) \\ &\quad - \psi\mathcal{B}\left(S_\varepsilon[((b_k)'_+(\varrho)\varrho - b_k(\varrho))\operatorname{div}\mathbf{u}] - \frac{1}{|\Omega|}\int_\Omega S_\varepsilon[((b_k)'_+(\varrho)\varrho - b_k(\varrho)\operatorname{div}\mathbf{u})]\,\mathrm{d}\mathbf{x}\right), \end{aligned} \tag{2.18}$$

furthermore,

$$\begin{aligned} \int_I \psi \int_\Omega a\varrho^\gamma S_\varepsilon[b_k(\varrho)]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \int_I \psi\left\{\int_\Omega a\varrho^\gamma\,\mathrm{d}\mathbf{x}\frac{1}{|\Omega|}\int_\Omega S_\varepsilon[b_k(\varrho)]\,\mathrm{d}\mathbf{x}\right\}\mathrm{d}t \\ &\quad + \mu\int_I\int_\Omega \sum_{i=1}^2 \partial_{x_j}u^i\partial_j\varphi^i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + (\mu+\lambda)\int_I\int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad - \int_I \psi\int_\Omega \sum_{i=1}^2 \varrho u^i\partial_t\phi^i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_I \psi'\int_\Omega \sum_{i=1}^2 \varrho u^i\phi^i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad - \int_I \psi\int_\Omega \sum_{1\leqslant i,j\leqslant 2} \varrho u^i u^j\partial_{x_j}\varphi_i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I\int_\Omega \sum_{1\leqslant i,j\leqslant 2} M^iM^j\partial_{x_j}\varphi^i\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad + \int_I\int_\Omega \frac{1}{2}|\mathbf{M}|^2\operatorname{div}\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_I \langle\varrho\nabla\Phi, \varphi\rangle\,\mathrm{d}t = \sum_{k=1}^9 J_k. \end{aligned} \tag{2.19}$$

Using Hölder inequality, Sobolev embedding theorem, the relations (2.19) and the property of $\mathcal{B}$ operator, thus the nine terms in the right hand of above relations can be estimated by the norm of (ϱ, $\mathbf{u}$, $\mathbf{M}$) and can be given by following estimations, please refer to [9,13,14] for details,

$$|J_1| = \left| \int_I \psi \left\{ \int_\Omega a\varrho^\gamma \, \mathrm{d}\mathbf{x} \frac{1}{|\Omega|} S_\varepsilon\big[b_k(\varrho)\big] \, \mathrm{d}\mathbf{x} \right\} \mathrm{d}t \right| \leqslant C \|\psi\|_{C(\bar{I})} \big\| S_\varepsilon\big[b_k(\varrho)\big] \big\|_{L^1(I' \times \Omega)},$$

$$|J_2| + |J_3| = \mu \left| \int_I \int_\Omega \partial_{x_j} u^i \partial_{x_j} \varphi^i \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \right| + \left| (\mu + \lambda) \int_I \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \right| \leqslant C \|\psi\|_{C(\bar{I})} \big\| S_\varepsilon\big[b_k(\varrho)\big] \big\|_{L^2(I' \times \Omega)}, \tag{2.20}$$

$$|J_4| = \left| \int_I \psi \int_\Omega \varrho u^i \partial_t \phi^i \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \right|$$
$$\leqslant C \|\psi\|_{C(\bar{I})} \big(\big\| S_\varepsilon\big[b_k(\varrho)\mathbf{u}\big] \big\|_{L^2(I', L^q(\Omega))} \big) + \big\| S_\varepsilon\big[\big((b_k)'_+(\varrho)\varrho - b_k(\varrho)\big) \operatorname{div} \mathbf{u}\big] \big\|_{L^2(I', L^r(\Omega))}, \tag{2.21}$$

$$|J_5| = \left| \int_I \psi' \int_\Omega \varrho u^i \phi^i \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \right| \leqslant C \big\| \psi' \big\|_{L^1(I)} \big\| S_\varepsilon\big[b_k(\varrho)\big] \big\|_{C(\bar{I}', L^p(\Omega))},$$

$$|J_6| = \left| \int_I \psi \int_\Omega \varrho u^i u^j \partial_{x_j} \varphi_i \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \right| \leqslant C \|\psi\|_{C(\bar{I})} \big\| S_\varepsilon\big[b_k(\varrho)\big] \big\|_{C(\bar{I}', L^p(\Omega))},$$

$$|J_7| + |J_8| = \left| \int_I \psi \int_\Omega M^i M^j \partial_{x_j} \varphi^i \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \right| + \left| \int_I \int_\Omega \psi \frac{1}{2} |\mathbf{M}|^2 \operatorname{div} \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \right| \leqslant C \|\psi\|_{C(\bar{I})} \big\| S_\varepsilon\big[b_k(\varrho)\big] \big\|_{C(\bar{I}', L^p(\Omega))}, \tag{2.22}$$

where $1 < q < \gamma/(\gamma - 1)$ ($\geqslant 4$, for $1 < \gamma \leqslant 4/3$), $1 < r = 2q/(q+2)$, $\theta < \theta p \leqslant \gamma$ and the constant C is independent of k and ε, and dependent on the norms $\|\varrho\|_{L^\infty(I, L^r(\Omega))}$, $\|\mathbf{u}\|_{L^2(I, W_0^{1,2}(\Omega))}$ and $\|\mathbf{M}\|_{L^2(I, W_0^{1,2}(\Omega))}$. As for the last term, from (1.17), we know

$$\sup_{t \in [0,T]} \big\| \varrho \nabla \Phi(t) \big\|_{W^{-1, \frac{\gamma}{2-\gamma}}(\Omega)} < \infty, \quad 1 < \gamma \leqslant 4/3,$$

due to the definition of norm $\|\cdot\|_{W^{-1, \frac{\gamma}{2-\gamma}}(\Omega)}$, we get

$$|J_9| = \int_I \langle \varrho \nabla \Phi, \varphi \rangle \, \mathrm{d}t$$
$$\leqslant \|\psi\|_{C(\bar{I})} \int_I \|\varrho \nabla \Phi\|_{W^{-1, \frac{\gamma}{2-\gamma}}(\Omega)} \left\| \mathcal{B} \left\{ S_\varepsilon\big[b(\varrho)\big] - \frac{1}{|\Omega|} \int_\Omega S_\varepsilon\big[b_k(\varrho)\big] \, \mathrm{d}\mathbf{x} \right\} \right\|_{W_0^{1, \frac{\gamma}{2\gamma-2}}(\Omega)} \mathrm{d}t$$
$$\leqslant C \|\psi\|_{C(\bar{I})} \big\| S_\varepsilon\big[b_k(\varrho)\big] \big\|_{C(\bar{I}', L^{\gamma/(2\gamma-2)}(\Omega))}, \tag{2.23}$$

where the constant C is independent of k and ε, and dependent on $\|\varrho \nabla \Phi\|_{L^\infty(I, W^{-1, \frac{\gamma}{2-\gamma}}(\Omega))}$. Taking $\varepsilon \to 0$ in (2.20)–(2.23), using [13, Lemma 6.5] on the property of time and space mollifiers, we can get

$$\begin{aligned}
&|J_1| \leqslant C \|\psi\|_{C(\bar{I})} \big\| b_k(\varrho) \big\|_{L^1(Q_T)}, \qquad |J_2| + |J_3| \leqslant C \|\psi\|_{C(\bar{I})} \big\| b_k(\varrho) \big\|_{L^2(Q_T)}, \\
&|J_4| \leqslant C \|\psi\|_{C(\bar{I})} \big(\big\| b_k(\varrho)\mathbf{u} \big\|_{L^2(I, L^q(\Omega))} + \big\| \big((b_k)'_+(\varrho)\varrho - b_k(\varrho)\big) \operatorname{div} \mathbf{u} \big\|_{L^2(I, L^r(\Omega))} \big), \\
&|J_5| \leqslant C \big\| \psi' \big\|_{L^1(I)} \big\| b_k(\varrho) \big\|_{L^\infty(I, L^p(\Omega))}, \\
&|J_6| \leqslant C \|\psi\|_{C(\bar{I})} \big\| b_k(\varrho) \big\|_{L^\infty(I, L^p(\Omega))}, \\
&|J_7| + |J_8| \leqslant C \|\psi\|_{C(\bar{I})} \big\| b_k(\varrho) \big\|_{L^\infty(I, L^p(\Omega))}, \\
&|J_9| \leqslant C \|\psi\|_{C(\bar{I})} \big\| b_k(\varrho) \big\|_{L^\infty(I, L^{\gamma/(2\gamma-2)}(\Omega))}.
\end{aligned} \tag{2.24}$$

By taking the supper limit in the left hand of (2.19), we have

$$\limsup_{\varepsilon \to 0} \int_I \psi \int_\Omega a\varrho^\gamma S_\varepsilon\big[b_k(\varrho)\big] \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = \int_I \psi \int_\Omega a\varrho^\gamma b_k(\varrho) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t.$$

By virtue of the definition of $b_k(t)$ and taking the proper value of q, for example $q = 3$, we can infer that, all the estimations in the right hand of $|J_1| - |J_9|$ of (2.24) can be controlled by a constant C which is independent of k and only dependent on the norms $\|\varrho\|_{L^\infty(I,L^r(\Omega))}$, $\|\mathbf{u}\|_{L^2(I,W_0^{1,2}(\Omega))}$ and $\|\mathbf{M}\|_{L^2(I,W_0^{1,2}(\Omega))}$, that is

$$\int_I \psi \int_\Omega \varrho^\gamma b_k(\varrho)\,\mathrm{d}x\,\mathrm{d}t \leqslant C\big(\|\psi'\|_{L^1(I)} + \|\psi\|_{C(\bar{I})}\big). \tag{2.25}$$

By monotonous convergence theorem, letting $k \to +\infty$ in (2.25), we get

$$\int_I \psi \int_\Omega \varrho^{\gamma+\theta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leqslant C\big(\|\psi'\|_{L^1(I)} + \|\psi\|_{C(\bar{I})}\big).$$

By the density approximation, we get

$$\int_{Q_T} \varrho^{\gamma+\theta}\,\mathrm{d}\mathbf{x} \leqslant C.$$

This completes the proof of Lemma 2.2. $\square$

Remark 2.3. By virtue of Lemma 2.2, the conclusions in Lemma 2.1 also hold for b satisfying (2.5)–(2.6), where $\theta(\gamma)$ takes $2\gamma - 1$.

Lemma 2.3. *If $\varrho \in L^{\gamma+\theta}(\Omega)$, $0 < \theta < \gamma - 1$ and $(9 + \sqrt{33})/12 < \gamma \leqslant 4/3$, then*

$$\langle \varrho\partial_{x_i}\Phi, \varphi\rangle = \int_\Omega \varrho\partial_{x_i}\Phi\varphi\,\mathrm{d}\mathbf{x}, \quad i = 1, 2. \tag{2.26}$$

Proof. The problem

$$\Delta\Phi = \varrho - \bar{\varrho}, \qquad \nabla\Phi\cdot\mathbf{n}|_{\partial\Omega} = 0 \quad \text{in the sense of trace}, \qquad \int_\Omega \Phi\,\mathrm{d}\mathbf{x} = 0, \tag{2.27}$$

has a unique solution Φ satisfying

$$\begin{aligned} &\Phi \in W^{2,\gamma+\theta}(\Omega), \qquad \nabla\Phi \in \big(L^{\frac{2(\gamma+\theta)}{2-(\theta+\gamma)}}(\Omega)\big)^2, \\ &\varrho\nabla\Phi \in \big(L^p(\Omega)\big)^2, \end{aligned} \tag{2.28}$$

where $p = 2\gamma(\theta+\gamma)/[(2-\gamma)(\theta+\gamma)+2\gamma]$ and θ satisfies $2\gamma(\theta+\gamma)/[(2-\gamma)(\theta+\gamma)+2\gamma] \geqslant 1$.

Let $\varepsilon > 0$ and $\varrho_\varepsilon \in \mathcal{D}(\Omega)$ satisfy $\|\varrho_\varepsilon - \varrho(t)\|_{L^{\gamma+\theta}(\Omega)} \to 0$ as $\varepsilon \to 0$, then there exists a unique Φ_ε to the problem

$$\Delta\Phi_\varepsilon = \varrho_\varepsilon - \bar{\varrho}_\varepsilon, \qquad \nabla\Phi_\varepsilon\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad \int_\Omega \Phi_\varepsilon\,\mathrm{d}\mathbf{x} = 0 \tag{2.29}$$

for the given ε. Let $\varphi \in \mathcal{D}(\Omega)$, using (2.29), we have

$$\langle \varrho_\varepsilon\partial_{x_i}\Phi_\varepsilon, \varphi\rangle := \int_\Omega \frac{1}{2}|\nabla\Phi_\varepsilon|^2\partial_{x_i}\varphi - \partial_{x_i}\Phi_\varepsilon\nabla\Phi_\varepsilon\nabla\varphi + \bar{\varrho}\partial_{x_i}\Phi_\varepsilon\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega \varrho_\varepsilon\partial_{x_i}\Phi_\varepsilon\varphi\,\mathrm{d}\mathbf{x}. \tag{2.30}$$

Make use of (2.27)–(2.29) and the property of mollifiers, we obtain

$$\begin{aligned} &\int_\Omega \varrho_\varepsilon\partial_{x_i}\Phi_\varepsilon\varphi\,\mathrm{d}\mathbf{x} \to \int_\Omega \varrho\partial_{x_i}\Phi\varphi\,\mathrm{d}\mathbf{x}, \qquad \int_\Omega \partial_{x_i}\Phi_\varepsilon\nabla\Phi_\varepsilon\nabla\varphi\,\mathrm{d}\mathbf{x} \to \int_\Omega \partial_{x_i}\Phi\nabla\Phi\nabla\varphi\,\mathrm{d}\mathbf{x}, \\ &\int_\Omega \frac{1}{2}|\nabla\Phi_\varepsilon|^2\partial_{x_i}\varphi\,\mathrm{d}\mathbf{x} \to \int_\Omega \frac{1}{2}|\nabla\Phi|^2\partial_{x_i}\varphi\,\mathrm{d}\mathbf{x}, \qquad \bar{\varrho}_\varepsilon\int_\Omega \partial_{x_i}\Phi_\varepsilon\varphi\,\mathrm{d}\mathbf{x} \to \bar{\varrho}\int_\Omega \partial_{x_i}\Phi\varphi\,\mathrm{d}\mathbf{x}. \end{aligned} \tag{2.31}$$

By (2.30)–(2.31), we infer

$$\langle \varrho_\varepsilon\partial_{x_i}\Phi_\varepsilon, \varphi\rangle \to \int_\Omega \frac{1}{2}|\nabla\Phi|^2\partial_{x_i}\varphi\,\mathrm{d}\mathbf{x} - \int_\Omega \frac{1}{2}\partial_{x_i}\Phi\nabla\Phi\nabla\varphi\,\mathrm{d}\mathbf{x} + \bar{\varrho}\int_\Omega \partial_{x_i}\Phi\varphi\,\mathrm{d}\mathbf{x} := \langle \varrho\partial_{x_i}\Phi, \varphi\rangle, \tag{2.32}$$

thus

$$\langle \varrho \partial_{x_i} \Phi, \varphi \rangle = \int_{\Omega} \varrho \partial_{x_i} \Phi \varphi \, \mathrm{d}\mathbf{x}.$$

This completes the proof of Lemma 2.3. □

Proof of Theorem 2.1. Combining Lemmas 2.2–2.3 with (2.3), we immediately get that (ϱ, Φ) satisfies

$$\int_0^T \langle \varrho \partial_{x_i} \Phi, \varphi \rangle \, \mathrm{d}t = \int_0^T \int_{\Omega} \varrho \partial_{x_i} \Phi \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \tag{2.33}$$

This yields Theorem 2.1. □

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